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On ($K_t - e$)-Saturated Graphs

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Abstract Given a graph H, we say a graph G is *H-saturated* if G does not contain H as a subgraph and the addition of any edge $e' \notin E(G)$ results in H as a subgraph. In this paper, we construct $(K_4 - e)$-saturated graphs with $|E(G)|$ either the size of a complete bipartite graph, a 3-partite graph, or in the interval $\left[2n-4, \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 6\right]$. We then extend the $(K_4 - e)$-saturated graphs to $(K_t - e)$-saturated graphs.

Keywords $K_4 - e \cdot K_t - e$ · saturated · edge spectrum

1 Introduction

The study of saturated graphs has seen a recent surge in popularity. A graph G is *H-saturated* if, given a graph H, G does not contain a copy of H but the addition of any edge $e \notin E(G)$ creates at least one copy of H within G. The question of the minimum number of edges of an H-saturated graph on n vertices, known as the *saturation number* and denoted $sat(n, H)$, has been addressed for many different types of graphs. The saturation number contrasts the popular question of the maximum number of edges possible in a graph G on n vertices that does not contain a copy of H, known as the Turán number and denoted $ex(n, H)$. Now, the topic of interest is the problem of finding the edge spectrum for H-saturated graphs. The *edge spectrum* of the family of H-saturated graphs on n vertices is the set of all possible sizes of an H-saturated graph. For terms not defined here see [4].

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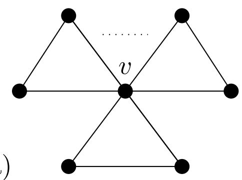
We are interested, then, in constructing graphs of size m that are H-saturated with $sat(n,H) \leq m \leq ex(n,H)$ and determining if such a graph exists for every possible value of m. This has been explored for few graphs H, including K_3, K_t and P_t. The spectrum for K_3-saturated graphs was found in 1995 by Barefoot, Casey, Fisher, Fraghnaugh, and Harary [3]. In [1], Amin, Faudree, and Gould found the spectrum for K_4-saturated graphs and in [2] Amin, Faudree, Gould and Sidorowicz found the spectrum for $K_t, t \geq 4$. Continuing this work, Gould, Tang, Wei, and Zhang addressed the edge spectrum of small paths [6].

One goal of this paper is to determine the edge spectrum of $(K_4 - e)$-saturated graphs, where $K_4 - e$ is the complete graph on four vertices with one edge removed. The graph $K_4 - e$ is isomorphic to the graph comprised of two triangles that share an edge, sometimes called a book. Further, $sat(n, K_4 - e) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$ (see [5]). The saturation number, $sat(n, K_4 - e)$, can be realized as the edge count of the graph on n vertices formed by $\frac{n-1}{2}$ triangles joined at a single vertex v when n is odd (Figure 1(a)) and $\frac{n-2}{2}$ triangles joined at the vertex v with an edge from v to the remaining vertex when n is even (Figure 1(b)). These graphs are $(K_4 - e)$-saturated as each vertex, except perhaps one, is a vertex of a triangle and an additional edge creates a second triangle with v, forming a copy of $(K_4 - e)$.

(a) 

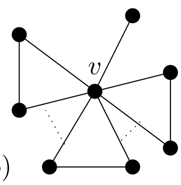
(b) 

Figure 1

The Turán number for an n vertex $(K_4 - e)$-free graph G is $ex(n, K_4 - e) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and can be realized by the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. The goal now is to construct graphs of size m that are $(K_4 - e)$-saturated with $\left\lfloor \frac{3(n-1)}{2} \right\rfloor \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$ and to determine if such a graph exists for every possible value of m.

2 Proof of Lower Bound

We begin with some useful lemmas.

Lemma 1 *If G is a connected $(K_4 - e)$-saturated graph, then $diam(G) = 2$.*

Proof Suppose that G is a connected $(K_4 - e)$-saturated graph. Let $x, y \in V(G)$ with $xy \notin E(G)$. Then $G + xy$ contains a $K_4 - e$ so there is a vertex $w \in V(G)$, distinct from x and y such that x, w, y is a path in G. Since this must be true for any pair of vertices in G, diam$(G) = 2$. ■

Lemma 2 *If G is a $(K_4 - e)$-saturated graph on n vertices with a cut vertex, then* $|E(G)| = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$.

Proof Let G be a $(K_4 - e)$-saturated graph with a cut vertex, say x. By Lemma 1, $\mathrm{diam}(G) = 2$ so every such path from u to v is of length 2, that is x is adjacent to every vertex $y \in V(G - x)$. Since it is possible to add an edge between two vertices of degree one without creating a copy of $(K_4 - e)$ and G is $(K_4 - e)$-saturated, there is a maximal matching in $V(G - x)$ that covers all except possibly one vertex. This creates $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ edge disjoint triangles, with one additional edge incident to x if n is even. This is precisely the graph that realizes the saturation number with an edge count of $|E(G)| = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$.■

Aside from the saturation number, small edge counts are not realizable by $(K_4 - e)$-saturated graphs. The following lemmas show the lower bound on the edge spectrum of $(K_4 - e)$-saturated graphs.

Lemma 3 *Let G be a connected (K_4-e)-saturated graph with minimum degree $\delta(G) \geq 3$ on $n \geq 10$ vertices. Then $|E(G)| \geq 2n - 4$.*

Proof Let G be a connected $(K_4 - e)$-saturated graph with minimum degree $\delta(G) \geq 3$. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n > 2n - 4$. Therefore there exists a vertex of degree exactly 3, say u. Note that $\mathrm{diam}(G) = 2$ by Lemma 1. Let u be adjacent to exactly three other vertices of G, say x, y and z. Let $X = \{x, y, z\}$ and let $A = V(G) - \{u, x, y, z\}$. Since $\mathrm{diam}(G) = 2$, every vertex in A is adjacent to at least one of the vertices in X. Let A_1 be the set of vertices in A that are adjacent to exactly one vertex of X, let A_2 be the vertices in A adjacent to exactly two vertices of X and let A_3 be the vertices in A adjacent to all vertices of X. The minimum degree condition implies that each $v \in A_1$ must be adjacent to at least two other vertices in A and each $w \in A_2$ must be adjacent to at least one other vertex in A. So we have a minimum edge count as follows:

$$\begin{aligned}
|E(G)| &\geq 3 + |A_1| + 2|A_2| + 3|A_3| + \left\lceil \frac{2|A_1|+|A_2|}{2} \right\rceil \\
&\geq 3 + 2|A_1| + 2|A_2| + \left\lceil \frac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 3 + 2(n - |A_3| - 4) + \left\lceil \frac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 2n - 5 + \left\lceil \frac{|A_2|}{2} \right\rceil + |A_3|.
\end{aligned}$$

If either A_2 or A_3 is non-empty, we are done. Thus, assume that $|A_2| = |A_3| = 0$. Then $|E(G)| \geq 2n - 5$ and it remains to show that there is at least one additional edge in G.

If at least one of the edges xy, yz, xz is in $E(G)$, we are done. Assume that xy, yz, and xz are not edges of G. Since $\delta(G) = 3$, there must be at least two vertices of A_1 adjacent to x, two vertices of A_1 adjacent to y and two vertices of A_1 adjacent to z. Then each vertex adjacent to x must be adjacent to at least one vertex adjacent to y and at least one vertex adjacent to z. Each vertex adjacent to y must be adjacent to at least one vertex adjacent to x and one vertex adjacent to z. Each vertex adjacent to z must be adjacent to at least one vertex adjacent to x and one vertex adjacent to y. This requirement allows the minimum possible edge count to remain at $|E(G)| \geq 2n-5$ as it requires at least $|A_1|$ edges amongst the vertices of A_1. However, this graph is not (K_4-e)-saturated, as adding xy does not create a copy of K_4-e, so there must be at least one additional edge. This completes the proof of the lemma. ■

Lemma 4 *Let G be a 2-connected (K_4-e)-saturated graph on m edges and $n \geq 10$ vertices. Then $m \geq 2n-4$.*

Proof Let G be a (K_4-e)-saturated, 2-connected graph on m edges. Since G is (K_4-e)-saturated, $\operatorname{diam}(G)=2$ by Lemma 1 and it follows from Lemma 3, that $m \geq 2n-4$ if $\delta(G) \geq 3$. Suppose $\delta(G)=2$ with $\deg(z)=2$ for some $z \in V(G)$. Then z is adjacent to some $x, y \in V(G)$ and we can partition the remaining vertices of G into three sets A, B, C with $A \subseteq N(x)$, $B \subseteq N(x) \cap N(y)$ and $C \subseteq N(y)$, (see Figure 2). Since G is 2-connected, A and B cannot both be empty, as this would make y a cut vertex. Similarly, C and B cannot both be empty. Note that if $B \neq \emptyset$ the edge from x to y is not in $E(G)$ as it would create a copy of K_4-e and for similar reasons, B must be an independent set.

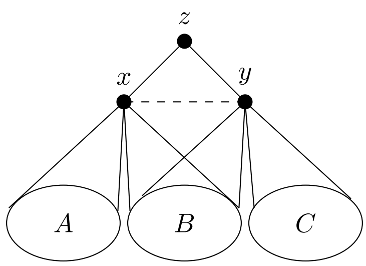


Figure 2

Case 1: Suppose both A and C are empty and B is not empty.
Each vertex in B is adjacent to both x and y, which creates a copy of C_4 for each vertex of B with the vertices x, y and z. Adding the edge xy, any edge between vertices of B, or the edge vz for some $v \in B$ will create a K_4-e, so the graph G is (K_4-e)-saturated. In this case, $m = 2+2(n-3) = 2n-4$.

Case 2: Suppose that A is empty and B, C are non-empty.
Since $\operatorname{diam}(G)=2$, there must be a path of length two from x to each $w \in C$ hence, there must be an edge from at least one $v \in B$ to each $w \in C$. Since G cannot contain a copy of K_4-e, each $w \in C$ must be adjacent to a distinct

vertex in B and hence $|B| \geq |C|$. Then each $w \in C$ is in a distinct triangle and is not adjacent to another vertex in C or a copy of $K_4 - e$ would exist in G. Additional edges will increase the edge count so $|E(G)|$ must be at least

$$\begin{aligned} m &\geq 2 + 2|B| + 2|C| \\ &= 2 + 2(n - |C| - 3) + 2|C| \\ &= 2n - 2|C| + 2|C| + 2 - 6 \\ &= 2n - 4. \end{aligned}$$

Note that by symmetry a similar argument holds when C is empty and A is nonempty.

Case 3: Suppose that A and C are both non-empty with $1 \leq |C| \leq |A|$ and B is empty.
Since G is $(K_4 - e)$-saturated, xy must be an edge of G and there can be no path of length 2 or more between any two vertices in A or between any two vertices in C. Also, $\operatorname{diam}(G) = 2$ implies that there is a $u - w$ path of length 1 or 2 for each $u \in A$ and each $w \in C$. So each $u \in A$ must be adjacent to at least $\left\lceil \frac{|C|}{2} \right\rceil$ vertices of C. There must also be at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ additional edges, either within C in the form of a matching, or between A and C if there is a vertex of A that is not in an edge in A.

If $|C| = 1$, $\operatorname{diam}(G) = 2$ requires that either $w \in C$ is adjacent to all vertices in A or $w \in C$ is adjacent to $\left\lceil \frac{|A|}{2} \right\rceil$ vertices in A and there are $\left\lfloor \frac{|A|}{2} \right\rfloor$ edges within A. In either case, $|E(G)| \geq 4+|A|+|A| = 2(n-3)+4 = 2n-2 > 2n-4$. Otherwise, we have the following edge count for G:

$$\begin{aligned} m &\geq 3 + |A| + |C| + |A| \left\lceil \frac{|C|}{2} \right\rceil + \left\lfloor \frac{|C|}{2} \right\rfloor \\ &= n + \left\lceil \frac{|C|}{2} \right\rceil \left(n - \left\lceil \frac{|C|}{2} \right\rceil - \left\lfloor \frac{|C|}{2} \right\rfloor - 3\right) + \left\lfloor \frac{|C|}{2} \right\rfloor \\ &= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right) n - \left\lceil \frac{|C|}{2} \right\rceil^2 - 3 \left\lceil \frac{|C|}{2} \right\rceil + 4 - \left\lfloor \frac{|C|}{2} \right\rfloor \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right) \\ &= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right) - \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right)\left(\left\lceil \frac{|C|}{2} \right\rceil + 4\right) - \left\lfloor \frac{|C|}{2} \right\rfloor \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right) \\ &= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1\right)(n - |C| - 4). \end{aligned}$$

Since $\left\lceil \frac{|C|}{2} \right\rceil \geq 1$ and $|A| = n - |C| - 3 \geq 1$ clearly hold, it follows that $|E(G)| \geq 2n - 4$ is always true.

Case 4: Suppose that A, B and C are non-empty with $1 \leq |C| \leq |A|$.
Then $\operatorname{diam}(G) = 2$ implies that there must be a path of length at most 2 from each $u \in A$ to each $w \in C$. The vertices x and y cannot be adjacent as B is

non-empty, which results in at least one C_4 with $v \in B$, x, y and z. Also, the vertices in B must be independent as any edge between two vertices in B will result in a $K_4 - e$ with x and y. If some $u \in A$ is not adjacent to any vertex in A or B, then the edge uz does not create a copy of $K_4 - e$, hence the graph is not $(K_4 - e)$-saturated. So if a vertex $u \in A$ is independent within A, then $uv \in E(G)$ for some $v \in B$ and either uw or vw is an edge of G for every $w \in C$. If a vertex $u \in A$ is not independent within A, then $uw \in E(G)$ for some $w \in C$ as an edge from A to B gives a $K_4 - e$. By symmetry, the same is true for vertices in C.

If $|C| = 1$, $\mathrm{diam}(G) = 2$ requires that there is a path of length 1 or 2 between $w \in C$ and each vertex in A. Since G is $(K_4 - e)$-saturated, wv is an edge in G for some $v \in B$ and either vu or wu is also an edge of G for some $u \in A$. Then $w \in C$ is adjacent to at least $\left\lceil \frac{|A|-1}{2} \right\rceil$ vertices in A and there are at most $\left\lfloor \frac{|A|-1}{2} \right\rfloor$ edges within A or from A to B. In any case, $|E(G)| \geq 3 + |A| + 2|B| + 1 + |A| = 2(n-4) + 4 = 2n - 4$.

Otherwise, each $u \in A$ must be adjacent to at least $\left\lceil \frac{|C|}{2} \right\rceil$ vertices of C. Then there must also be at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ additional edges, either within C in the form of a matching, or between A and C if there is a vertex of A that is not in an edge in A. This yields the following edge count of G:

$$
\begin{aligned}
m &\geq 2 + |A| + 2|B| + |C| + |A| \left\lceil \tfrac{|C|}{2} \right\rceil + \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= n - 1 + |B| + (n - |B| - |C| - 3) \left\lceil \tfrac{|C|}{2} \right\rceil + \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= 2n - 1 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (n - |B|) - \left\lceil \tfrac{|C|}{2} \right\rceil^2 - 3 \left\lceil \tfrac{|C|}{2} \right\rceil - \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left(n - |B| - \left\lfloor \tfrac{|C|}{2} \right\rfloor \right) - \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left(\left\lceil \tfrac{|C|}{2} \right\rceil + 4 \right) \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (n - |B| - |C| - 4). \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (|A| - 1).
\end{aligned}
$$

So $|E(G)| \geq 2n - 4$ if $|C| \geq 3$ and $|A| \geq 2$. However, if $|A| = 1$ then $|E(G)| \geq 2n - 4$, similar to the case when $|C| = 1$, so it remains to determine the edge count of G when $|A| = |C| = 2$.

Suppose that $C = \{w, w'\}$. If $ww' \in E(G)$, wv and $w'v$ are not edges of G for any $v \in B$. Then $\mathrm{diam}(G) = 2$ implies that each $u \in A$ is adjacent to w or w'. Also, since G is $(K_4 - e)$-saturated, either there is an edge in A or there is an edge from A to B as adding one of those edges does not create a copy of $K_4 - e$. This yields $|E(G)| \geq 6 + 2|B| + 2 + 1 + 1 = 10 + 2(n-7) = 2n - 4$. On the other hand, if $ww' \notin E(G)$, both wv and $w'v'$ are edges of G for distinct $v \in B$

and $v' \in B$ without creating a copy of $K_4 - e$ and each vertex of C must be adjacent to at least one vertex in A such that $|E(G)| \geq 6+2|B|+2+2 = 2n-4$.

This completes the proof of the lemma. ■

3 Proof of Theorem

We will now show that there is a ($\mathrm{K}_4 - e$)-saturated graph for every integer value of m in the interval $[2n-4, \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 6]$ by combining two different constructions.

Theorem 1 *There exists a $(K_4 - e)$-saturated graph on $n \geq 6$ vertices and m edges where $2n - 4 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 6$.*

Proof Case 1: Suppose $2n - 4 \leq m \leq 3n - 9$.

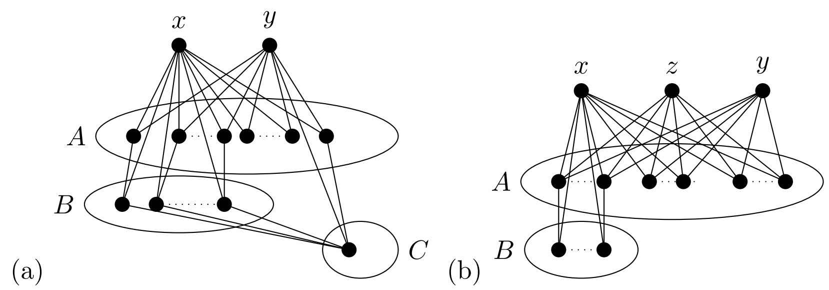


Figure 3

To construct $(K_4 - e)$-saturated graphs we modify $K_{2,n-2}$ and $K_{3,n-3}$. For the graph in Figure 3(a), we form a set B by removing at most $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices from A so that the vertices are adjacent to x and to distinct vertices in A and we form a set C with a single vertex from A so that it is adjacent to y and a vertex in A that is not adjacent to any vertex in B. We then join each vertex of B to the vertex in C.

We now show the resulting graphs are $(K_4 - e)$-saturated. First, the edge xy will create at least two triangles on that edge if $|A| \geq 2$. Any edge added within A will create a $K_4 - e$ with x and y and any edge added within B (or within C) will create a triangle with x (or y, respectively), which creates a copy of $K_4 - e$ as every pair of vertices in B (or C) is a part of two triangles. Each edge from A to B is an edge of a triangle with x so any additional edge between $u \in A$ and $v \in B$ will create another triangle with x and the edge xv, resulting in a copy of $(K_4 - e)$. Similarly, an edge between any vertex in A and $w \in C$ with create a $K_4 - e$. Finally, any additional edge from x to $w \in C$ or y to $v \in B$ will create a $(K_4 - e)$ with the triangle constructed between x or y and B or C, sharing the edge from A to w or v, respectively.

For the graph in Figure 3(a), if $|A| = n - b - 3$ where $|B| = b$ and $|C| = 1$, the edge count is:

$$\begin{aligned} m &= 2|A| + 3|B| + 2 \\ &= 2(n - b - 3) + 3b + 2 \\ &= 2n - 4 + b. \end{aligned}$$

So we have an edge count of $m = 2n - 4 + b$, which increases by one as the size of B increases by one. Since $0 \leq |B| \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 1$, we have $2n - 4 \leq m \leq 2n - 4 + \left\lfloor \frac{n-2}{2} \right\rfloor - 1 = \left\lfloor \frac{5n}{2} \right\rfloor - 6$.

For the graph in Figure 3(b), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A. Similar to the graphs in Figure 3(a), such graphs are $(K_4 - e)$-saturated and, if $|A| = n - b - 3$ where $|B| = b$, they have edge count:

$$\begin{aligned} m &= 3|A| + 2|B| \\ &= 3(n - b - 3) + 2b \\ &= 3n - 9 - b. \end{aligned}$$

So the edge count decreases by one as the size of B increases by one. Since $0 \leq |B| \leq \left\lfloor \frac{n-3}{2} \right\rfloor$, we have $3n - 9 \geq m \geq 3n - 9 - \left\lfloor \frac{n-3}{2} \right\rfloor = \left\lfloor \frac{5n}{2} \right\rfloor - 7$, which clearly intersects the interval constructed with the graphs of Figure 3(a). Thus, we have constructed saturated graphs of size $2n - 4$ to $3n - 9$ for $n \geq 6$ and this case completes the proof of Theorem 1 for $n \leq 11$.

Case 2: Suppose $3n - 9 \leq m \leq 4n - 18$.

We can similarly modify the complete bipartite graphs $K_{3,n-3}$ and $K_{4,n-4}$ by adding triangles to the vertices in the smaller vertex set in the same way as before, to obtain a $(K_4 - e)$-saturated graph for $n \geq 11$.

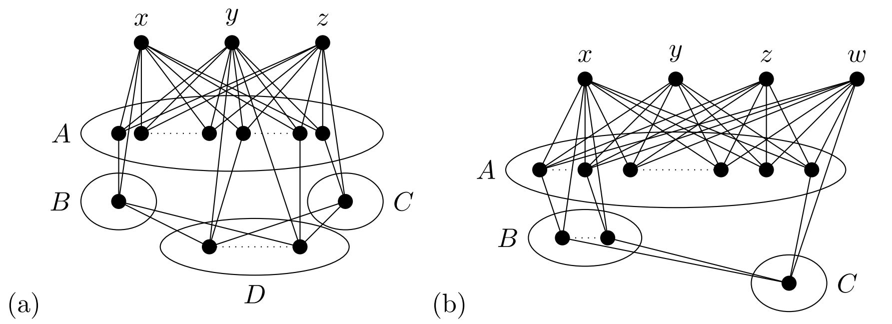


Figure 4

For the graph in Figure 4(a), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A. We form a set C with the vertices from A that are adjacent to z and distinct vertices in A and we a form set D with vertices from A that are adjacent to y and distinct vertices in A. In forming the sets B, C and D, it is necessary that their neighbors in A do not overlap. We then join each vertex of B and C to all of D. If $|A| = n - b - c - d - 3$ where $|B| = b$, $|C| = c$ and $|D| = d \geq 2$, then the edge count is:

$$\begin{aligned} m &= 3|A| + 2|B| + 2|C| + 2|D| + |B||D| + |C||D| \\ &= 3(n - b - c - d - 3) + 2b + 2c + (2 + b + c)d \\ &= 3n - b - c + (b + c - 1)d - 9. \end{aligned}$$

If we fix $|B| = |C| = 1$, we have an edge count of $m = 3n - 11 + d$, which increases by one as the size of D increases by one. Since the construction requires $2 \leq |B| + |C| + |D| \leq \left\lfloor \frac{n-3}{2} \right\rfloor$, we have $2 \leq |D| \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 2$ so that $3n - 9 \leq m \leq 3n - 11 + \left\lfloor \frac{n-3}{2} \right\rfloor - 2 = \left\lfloor \frac{7n}{2} \right\rfloor - 13$.

For the graph in Figure 4(b), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A, we form a set C with the vertices from A that are adjacent to w and distinct vertices in A. Then we join each vertex in C to all vertices in B. If $|A| = n - b - c - 4$ where $|B| = b$ and $|C| = c$, then the edge count is:

$$\begin{aligned} m &= 4|A| + 2|B| + 2|C| + |B||C| \\ &= 4(n - b - c - 4) + 2b + 2c + bc \\ &= 4n - 2b - 2c + bc - 16. \end{aligned}$$

Thus, if we fix $|C| = c = 1$, we have an edge count of $m = 4n - 18 - b$, which decreases by one as the size of B increases by one. Then $0 \leq |B| \leq \left\lfloor \frac{n-4}{2} \right\rfloor - 1$ implies $4n - 18 \geq m \geq 4n - 18 - \left(\left\lfloor \frac{n-4}{2} \right\rfloor - 1\right) = \left\lfloor \frac{7n}{2} \right\rfloor - 15$, which intersects the interval for the graphs of Figure 4(a). Thus, we have constructed graphs of size $3n - 9$ to $4n - 18$ for $n \geq 11$.

Case 3: Suppose $4n - 18 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 5$.

We blow-up the graph C_5 such that each vertex becomes a set of independent vertices with adjacencies according to the original C_5, where an edge $xy \in E(C_5)$ becomes a $K_{s,t}$, when $x \in V(C_5)$ blows-up to a set of s vertices and $y \in V(C_5)$ blows-up to a set of t vertices. Then any edge added within a set of independent vertices will create at least two triangles on that edge with vertices of two adjacent sets. Also, any edge added between vertices in two different vertex sets will create at least two triangles on that edge with vertices of the common adjacent set, if the common adjacent set has order at least 2. As such, a blown-up C_5 in Figure 5(a), with at least two vertices in each vertex set, is $(K_4 - e)$-saturated.

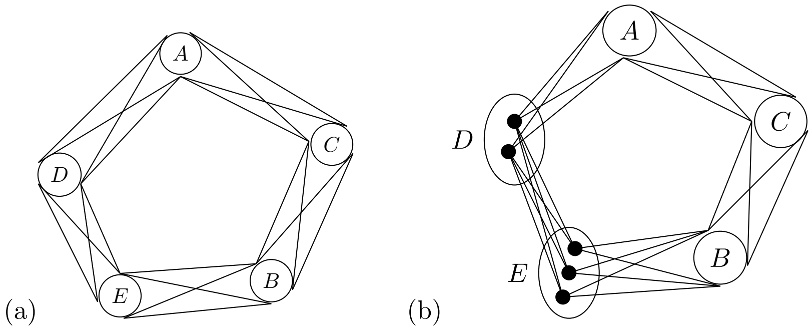


Figure 5

The blown-up C_5 in Figure 5(b), which we denote as $G = C_5[A, D, E, B, C]$, is $(\mathrm{K}_4 - e)$-saturated with $|A| = n - b - c - 5$ provided $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$ with $|E(G)| = m$ given by the products of the orders of consecutive vertex sets, hence:

$$\begin{aligned} m &= |A||D| + |D||E| + |E||B| + |B||C| + |C||A| \\ &= 2(n - b - c - 5) + 2(3) + 3b + bc + c(n - b - c - 5) \\ &= cn + 2n - c^2 - 7c + b - 4 \\ &= (n - c)(c + 2) - 5c + b - 4. \end{aligned}$$

Then for fixed values of c, when b increases by 1, that is, as vertices are moved from A to B, the edge count increases by 1. To maintain at least two vertices in each set of the blown-up C_5, we must have $b \in [2, n - c - 7]$. If we let $b = n - c - 7$ for fixed n, then we have $m = cn + 3n - c^2 - 8c - 11$, which is maximized when $c = \left\lceil \frac{n}{2} \right\rceil - 4$ such that $c \in [2, \left\lceil \frac{n}{2} \right\rceil - 4]$. Then the smallest edge count for G is when $a = n - 9, b = 2, c = 2$ and is $m = (n-2)(4) - 10 + 2 - 4 = 4n - 20 \leq 4n - 18$ and the largest possible edge count is given when $a = 2, b = \left\lfloor \frac{n}{2} \right\rfloor - 3, c = \left\lceil \frac{n}{2} \right\rceil - 4$ and is:

$$\begin{aligned} m &= \left(n - \left\lceil \tfrac{n}{2} \right\rceil + 4\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 4 + 2\right) - 5\left(\left\lceil \tfrac{n}{2} \right\rceil - 4\right) + \left(\left\lfloor \tfrac{n}{2} \right\rfloor - 3\right) - 4 \\ &= \left(\left\lfloor \tfrac{n}{2} \right\rfloor + 4\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 2\right) - 5\left\lceil \tfrac{n}{2} \right\rceil + 20 + \left\lfloor \tfrac{n}{2} \right\rfloor - 3 - 4 \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - 2\left\lfloor \tfrac{n}{2} \right\rfloor + 4\left\lceil \tfrac{n}{2} \right\rceil - 8 - 5\left\lceil \tfrac{n}{2} \right\rceil + 13 + \left\lfloor \tfrac{n}{2} \right\rfloor \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - \left\lfloor \tfrac{n}{2} \right\rfloor - \left\lceil \tfrac{n}{2} \right\rceil + 5 \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - n + 5. \end{aligned}$$

Next, we check that the entire interval $\left[4n - 18, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 5\right]$ of length $\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - 5n + 24$ is covered. For each fixed c, we will have an interval of values S_c determined by the range of values for b, namely, each interval has a left endpoint given when $b = 2$ such that the interval starts at $m = (n - c)(c + 2) - 5c - 2$. So we have a $(K_4 - e)$-saturated graph on n vertices and m

edges for an interval of length $(n-c-7)-2+1 = n-c-8$ and we have $\left(\left\lceil \frac{n}{2} \right\rceil - 4\right) - 2 + 1 = \left\lceil \frac{n}{2} \right\rceil - 5$ such intervals. Then the next consecutive interval will start at:

$$\begin{aligned} m &= (n-(c+1))(c+1+2) - 5(c+1) - 2 \\ &= (n-c)(c+2) + (n-c) - (c+2) - 1 - 5c - 7 \\ &= (n-c)(c+2) - 5c - 2 + (n-2c-8). \end{aligned}$$

Thus, the end of each interval S_c will overlap with the next interval S_{c+1} in the first $(n-c-8)-(n-2c-8) = c$ numbers. There are $\left\lceil \frac{n}{2} \right\rceil - 5$ intervals, each with $(n-c-8)-(c)+1 = n-2c-7$ distinct elements. As the lowest interval starts at $4n-18$ and the largest interval ends at $\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 5$ with each interval having a nonempty overlap with the next highest interval, all values are covered.

Case 4: Suppose $m = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 6$.

Similar to the first two cases, we modify the complete bipartite $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$. Let A be the first partite set and B the second partite set. We remove all edges from one vertex $v \in A$ and all edges from one vertex $v' \in B$. We add an edge from v to a vertex $a \in A$ and a vertex $b \in B$, creating a triangle. Similarly, we add an edge from v' to a vertex $a' \in A$ and a vertex $b' \in B$, where $a \neq a'$ and $b \neq b'$. Finally, we add the edge vv'. Since this is the same modification as in Case 1, the resulting graph is similarly $(K_4 - e)$-saturated with an edge count of $m = \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 5 = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 6$.

This completes the proof of the theorem. ■

4 Graphs in $\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 7, \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil\right]$

We conjecture that graphs with sizes in the interval $\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 7, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor\right]$ are of two types: complete bipartite graphs with partite sets of nearly equal size, and 3-partite graphs with two partite sets of nearly equal size and one partite set of order one. The complete bipartite graph is $(K_4 - e)$-saturated as adding an edge between any two nonadjacent vertices will create a $K_4 - e$. In the 3-partite graph, we let the two larger partite sets induce a complete bipartite graph and the single vertex set be adjacent to exactly one vertex in each of the other partite sets. This 3-partite graph, then, contains a complete bipartite graph on $n-1$ vertices as well as a single triangle and is $(K_4 - e)$-saturated. If an edge is added within either of the independent sets a copy of $K_4 - e$ is created and if an edge is added between a vertex of the triangle and any other vertex of the graph, a $K_4 - e$ is created. Such graphs would have the highest possible edge counts when the larger partite sets are almost the same order.

Let the graph G be a complete bipartite graph with one partite set of order $\left\lfloor \frac{n}{2} \right\rfloor + k$. Then the size of G is $m = \left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)\left(\left\lceil \frac{n}{2} \right\rceil - k\right)$. This gives a few additional values of m in the interval $\left[\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - n + 7, \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right]$. Let the graph H be a 3-partite graph described above with partite sets of order $\lceil\frac{n}{2}\rceil + k$, $\lfloor\frac{n}{2}\rfloor - k - 1$, and 1. Then the size of H is:

$$\begin{aligned} m &= \left(\lceil\tfrac{n}{2}\rceil + k\right)\left(\lfloor\tfrac{n}{2}\rfloor - k - 1\right) + 2 \\ &= \lceil\tfrac{n}{2}\rceil\lfloor\tfrac{n}{2}\rfloor - k\lceil\tfrac{n}{2}\rceil - \lceil\tfrac{n}{2}\rceil + k\lfloor\tfrac{n}{2}\rfloor - k^2 - k + 2. \end{aligned}$$

Which is $\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - \lceil\frac{n}{2}\rceil - k^2 - k + 2$ for n even and $\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - \lceil\frac{n}{2}\rceil - k^2 - 2k + 2$ for n odd. For small values of k, this gives additional values of $m \in \left[\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - n + 7, \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - \lceil\frac{n}{2}\rceil + 2\right]$. We believe that this completes the edge spectrum of $(K_4 - e)$-saturated graphs since the graphs G and H are completely saturated with four vertex complete graphs missing two edges.

Thus, the edge spectrum of $(K_4 - e)$-saturated graphs has a jump from the saturation number to the next possible edge count, is continuous in the interval $\left[2n - 4, \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - n + 6\right]$ and then,we believe, has sporadic values in the interval $\left[\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor - n + 7, \lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right]$.

5 Constructing $(K_t - e)$-saturated Graphs

Given a graph G that is $(K_4 - e)$-saturated, it is possible to construct a graph $G' = G + v$ that is $(K_5 - e)$-saturated where $G + v$ is constructed by adding a vertex v and all edges from v to each vertex in G. Then, by joining a vertex to each of the $(K_4 - e)$-saturated graphs we have constructed, there is a $(K_5 - e)$-saturated graph on n vertices for each edge count in

$$\begin{aligned} &\left[2(n-1) - 4 + (n-1), \lfloor\tfrac{n-1}{2}\rfloor\lceil\tfrac{n-1}{2}\rceil - (n-1) + 6 + (n-1)\right] \\ &= \left[3n - 7, \lfloor\tfrac{n-1}{2}\rfloor\lceil\tfrac{n-1}{2}\rceil + 6\right]. \end{aligned}$$

There are also $(K_5 - e)$-saturated graphs for sporadic values of m in the interval $\left[\lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil + 7, \lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil + n - 1\right]$.

Similarly, given a graph H that is $(K_{t-1} - e)$-saturated, it is possible to construct a graph $H' = H + v$ that is $(K_t - e)$-saturated where $H + v$ is constructed by adding a vertex v and all edges from v to each vertex in H. So joining $t - 4$ vertices, one at a time, to a $(K_4 - e)$-saturated graph on $n - t + 4$ vertices will result in a $(K_t - e)$-saturated graph. As such, there is a $(K_t - e)$-saturated graph on n vertices and m edges for each value in the interval

$$\begin{aligned} &\left[2n - 2t + 4 + \textstyle\sum_{i=1}^{t-4}(n - t + 3 + i), \lfloor\tfrac{n-t+4}{2}\rfloor\lceil\tfrac{n-t+4}{2}\rceil - n + t + 2 + \sum_{i=1}^{t-4}(n - t + 3 + i)\right] \\ &= \left[(t-2)n - \tfrac{t^2}{2} + \tfrac{3}{2}t - 2, \left(\lfloor\tfrac{n-t}{2}\rfloor + 2\right)\left(\lceil\tfrac{n-t}{2}\rceil + 2\right) - (n-t) + n(t-4) - \tfrac{t^2}{2} + \tfrac{7}{2}t - 4\right] \\ &= \left[(t-2)n - \tbinom{t-1}{2} - 1, \lfloor\tfrac{n-t}{2}\rfloor\lceil\tfrac{n-t}{2}\rceil + (t-3)n - \tbinom{t-2}{2} - 1\right]. \end{aligned}$$

Also, there are $(K_t - e)$-saturated graphs for sporadic values of m in

$$\left[\left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2} + 4, \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-2)n - \binom{t-1}{2} - 1\right].$$

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